UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1962

PART I

for Internal and External Students

(1) MATHEMATICS I

Tuesday, 19 June: 10 to 1

Full marks may be obtained for correct answers to about SEVEN questions.

1. (a) If

$$y = \frac{\sin^{-1} x}{\sqrt{(1 - x^2)}},$$

prove that

$$(1 - x^2)\frac{dy}{dx} = xy + 1.$$

(b) ~~Find dy/dx in its simplest form if~~

$$y = \frac{(2x - 1)(3 - x)^3}{(2 - x)^2}.$$

(c) If $x = \log \tan \frac{1}{2}\theta$, $y = \tan \theta - \theta$, prove that

$$\frac{d^2 y}{dx^2} = \tan^2 \theta \sin \theta (\cos \theta + 2 \sec \theta).$$

2. (a) Expand

$$\sqrt{(1 + ax + bx^2 + cx^3)} - \sqrt{(1 + ax + bx^2)}$$

in ascending powers of x as far as the term in x^5.

(b) Write down the expansion in ascending powers of x of e^x and $\log(1 + x)$ and state the range of values of x for which each expansion is valid.
Find the limiting value, as x tends to zero, of

$$\frac{\log\{\frac{1}{2}(e^x + e^{-x})\} - \frac{1}{2}x^2}{x^4}.$$

Turn over

3. Obtain the equation of the normal at the point $P(a\cos\theta,\ b\sin\theta)$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and prove that the length of the perpendicular p from the origin to this normal is

$$\frac{a^2 - b^2}{\sqrt{(a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta)}}.$$

Show that the greatest value of p is given by $\tan^2\theta = b/a$.

4. If $f(x) = Ax^2 + Bx + C$, where A, B and C are constants, prove that

$$\int_a^b f(x)\,dx = \frac{1}{6}(b-a)\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}.$$

Deduce Simpson's rule for the approximate evaluation of a definite integral. Apply the rule to obtain an approximate value of

$$\int_1^2 (x-1)^2 \log_{10} x\,dx,$$

by dividing up the range of integration into 10 equal parts.

5. (a) Find the loci in the Argand diagram corresponding to the equations

$$\text{(i)}\quad \arg\frac{z-1}{z-j} = \frac{\pi}{2}, \qquad \text{(ii)}\quad |z-1| = 2|z-j|.$$

(b) If the point $z = x + jy$ describes the circle $|z - 1| = 1$, show that the real part of $1/(z-2)$ is constant.

6. Define $\cosh x$ and $\tanh x$ and show that $\cosh x \geqq 1$ and $-1 < \tanh x < 1$. If x and y satisfy the equations

$$\cosh x \cosh y = 2, \qquad \sinh x \sinh y = -1,$$

show by direct solution that $x = -y = \pm\log(1 + \sqrt{2})$.

7. Show that the coordinates of the centre of curvature at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ are $\{a(2 + 3t^2),\ -2at^3\}$. Find the coordinates of the points where the locus of the centres of curvature meets the parabola.

8. (a) Solve the differential equation

$$x\frac{dy}{dx} = y\log x.$$

(b) Solve the differential equation

$$\frac{dy}{dx} + \frac{y^2 + y + 1}{(x+1)^2} = 0$$

and find the equation of the family of curves any one of which is orthogonal to a solution of this differential equation.

9. A train of mass m lb moves on straight horizontal rails against a resistance kv lb wt where v ft/sec is the velocity and k is a constant. The engine works at a constant rate of $400k$ ft lb/sec. Show that the time taken to increase the speed of the train from 4 ft/sec to 10 ft/sec is $(m/2kg) \log \frac{32}{25}$ sec and find the distance covered in this time interval.

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