4073

UNIVERSITY OF LONDON

GENERAL CERTIFICATE OF EDUCATION EXAMINATION

Advanced Level

SUMMER, 1960

PURE MATHEMATICS.—II

TUESDAY, June 21.—*Afternoon,* 2 *to* 5

Not more than EIGHT *questions are to be attempted.*

1. (i) If the coefficients of x and x^2 in the expansion of

$$(1 + x + ax^2)(1 + bx)^{10}$$

are both zero, find the numerical values of a and b and the coefficient of x^3.

(ii) Solve the equation

$$2 + \log_{10} x = \log_e x .$$

2. (i) Prove that $\tan 75^\circ = 2 + \sqrt{3}$ and show that $\tan 37^\circ\, 30'$ will be the positive root of the equation

$$t^2 + 2(2 - \sqrt{3})\,t - 1 = 0 .$$

Account for the negative root.

(ii) Prove that, for all values of the angles α and β,

(*a*) $\sin \alpha \cos 2\beta - 2 \sin(\alpha - \beta) \cos \beta = \cos \alpha \sin 2\beta - \sin \alpha$,

(*b*) $\tan 4\alpha - \tan 3\alpha - \tan \alpha = \tan 4\alpha \tan 3\alpha \tan \alpha$.

If $\tan \alpha - \tan \beta - \tan \gamma = \tan \alpha \tan \beta \tan \gamma$, deduce a relationship between α, β and γ.

3. A is a listening post a miles due N of another post B. A signal reaches A from the direction 135° (S 45° E) and reaches B from the direction 045° (N 45° E). If these directions are accurate find the distance from A of the origin of the signal.

If either direction may be as much as $\alpha°$ in error, indicate on a sketch the region from which the signal must have come, and prove that the area of this region is

$$\tfrac{1}{2}a^2 \sin 2\alpha \tan 2\alpha \; .$$

4. $VABC$ is a tetrahedron every edge of which is of length a. It is placed with its base ABC upon a horizontal table. Find the height of V above the table.

P and Q are the mid-points of VA and VB respectively. Find the area of (i) the section of the tetrahedron containing the points P, Q and C, (ii) the vertical section of the tetrahedron which passes through P and Q.

5. O is the origin, P any point on the part of the line $x + y = a$ which lies in the first quadrant, N the foot of the perpendicular from P upon the x-axis. Find the equation of the locus of (i) the circumcentre and (ii) the centroid of the triangle OPN.

Prove that ordinates P_1N_1 and P_2N_2 can be found such that the circumcentre of the triangle OP_1N_1 coincides with the centroid of the triangle OP_2N_2. Give the coordinates of P_1 and P_2 and show that P_2 lies on the circumcircle of the triangle OP_1N_1.

6. P is the point $(a, 2a)$ on the parabola $y^2 = 4ax$. Find the equation of the normal to the parabola at P and verify that it passes through the point $(5a, -2a)$.

Prove that every chord of the parabola, other than this normal, which passes through $(5a, -2a)$ subtends a right angle at P.

7. Write down (without proof) the equations of the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a \cos \theta, b \sin \theta)$ and to the hyperbola $4xy = ab$ at the point (x', y').

Find the four values of θ for which the point $(a \cos \theta, b \sin \theta)$ lies on the hyperbola $4xy = ab$.

Prove that the ellipse and the hyperbola can only intersect each other at an angle of 60° in the special case in which the ellipse becomes a circle.

8. (i) Find for what range of values of x the function $e^{-x}(x^2 + 4x - 11)$ increases when x increases.

(ii) Cans in the form of right circular cylinders are to be manufactured from sheet metal. Some are to be open at one end and closed at the other and others are to be closed at both ends. Prove that, in order to obtain a maximum volume from a given area of sheet metal, the height of the cylinder should be made equal to the radius of the base in the one case and to the diameter of the base in the other.

9. The curve $2y = 7x - 2x^2$ is cut by the line $2y = x$ at the origin O and at the point A. P is a point on the arc of the curve between O and A. The ordinate through P cuts OA in Q. Find the position of P when PQ has its maximum length, and verify that in this position the tangent to the curve at P is parallel to OA.

Find the ratio of the areas into which PQ, when in this position, divides the area between the curve and OA.

10. (i) If $f'(x) = x^2(1 + x^3)^{10}$ and $f(0) = 1$, find $f''(x)$ and $f(x)$, the dashes denoting differentiations with respect to x.

(ii) Evaluate

$$\int_0^{\pi/2} \sin x \cos 4x \, dx \quad \text{and} \quad \int_0^{\pi/2} \sin^3 x \cos^4 x \, dx.$$