UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1962

PART I

for Internal and External Students

(1) MATHEMATICS II

Tuesday, 19 June: 2.30 to 5.30

Full marks may be obtained for correct answers to about SEVEN questions.

1. **(a)** Show that

$$\int_0^2 \frac{5x\,dx}{(x+1)(x^2+4)} = \frac{\pi}{2} - \frac{1}{2}\log\left(\frac{9}{2}\right).$$

(b) Evaluate

$$\text{(i)}\ \int_0^1 \sin^{-1} x\,dx, \qquad \text{(ii)}\ \int_{a/2}^{a} \frac{dx}{x\sqrt{(a^2-x^2)}}.$$

2. If $I_n = \int_0^{\pi/2} \cos^n x\,dx$ where $n > 1$, show that

$$I_n = \frac{n-1}{n} I_{n-2}.$$

Hence, or otherwise, calculate

(i) the mean value of $(a^2 - x^2)^{3/2}$ over the range $0 \leqq x \leqq a$,

(ii) the root mean square of $\cos^3\theta$ over the range $0 \leqq \theta \leqq \pi$.

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Turn over

3. If $y = \{x + \sqrt{(1 + x^2)}\}^{3/2}$, show that

(i) $4(1 + x^2)\left(\dfrac{dy}{dx}\right)^2 = 9y^2$,

(ii) $4(1 + x^2)\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - 9y = 0.$

Hence, or otherwise, obtain the power-series expansion of y in ascending powers of x as far as the term in x^3.

4. Show that the equation of the tangent at the point P with parameter t on the curve

$$x = 2a\cos^3 t, \qquad y = a\sin^3 t, \qquad (0 \leq t \leq \tfrac{1}{2}\pi)$$

is

$$x\sin t + 2y\cos t - 2a\sin t\cos t = 0$$

and obtain the equation of the normal to the curve at P.
If the tangent at P cuts the y-axis at A calculate the area of the triangle POA where O is the origin, and find the greatest value of this area.

5. (a) Find the three complex roots of the equation

$$z^3 = 1 - j$$

giving your answers in the form $r\,e^{j\theta}$ where r is positive and $-\pi < \theta \leq \pi$. Indicate the positions of the roots on an Argand diagram. Express the root lying in the second quadrant of the Argand diagram in the form $a + jb$, giving a and b correct to two places of decimals.

(b) Apply De Moivre's theorem to obtain the expansion of $\sin 7\theta$ as a polynominal in $\sin\theta$.

6. Sketch the curve $ay^2 = x^2(a - x)$, where $a > 0$, and show that the tangents at the origin to this curve are perpendicular. Calculate the area A enclosed by the x-axis and that part of the curve which lies in the first quadrant. Calculate also the volume swept out when the area A is rotated through 2π about (i) the x-axis, (ii) the y-axis.
Deduce the coordinates of the centroid of the area A.

7. Sketch the curve $r = a\cos^2\theta$, and find the area enclosed by that loop of the curve for which $-\pi/2 \leq \theta \leq \pi/2$.
Show that the length l of this loop is

$$2a\int_0^{\pi/2}\cos\theta\sqrt{(1 + 3\sin^2\theta)}\,d\theta$$

and by the substitution $\sin\theta = (1/\sqrt{3})\sinh x$ or otherwise, show that

$$l = \frac{a}{\sqrt{3}}\{2\sqrt{3} + \log(2 + \sqrt{3})\}.$$

8. (a) If $xy = C_1 \cos 2x + C_2 \sin 2x$, where C_1 and C_2 are arbitrary constants, obtain the second order differential equation satisfied by y in the form

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 0.$$

(b) Solve the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 36x$$

given that $y = 0$ and $dy/dx = -10$ when $x = 0$.
Show that in this case the graph of y against x has an inflexion where $x = -\log \frac{27}{2}$.

9. Prove that the moment of inertia of a uniform solid circular cylinder, of radius a, length $3a$ and mass M, about a diameter of one end is $13Ma^2/4$.
The cylinder can turn freely about a fixed horizontal axis which coincides with a diameter of one end. The cylinder is released from rest when its axis makes an angle β with the downward vertical. Calculate the angular velocity of the cylinder in the subsequent motion when the axis of the cylinder makes an angle θ with the downward vertical and deduce the period of small oscillations about the position of stable equilibrium.

A. GEARY
C. PLUMPTON

grad $y = x + c$

$2y \cos t = -x \sin t$

$\therefore y = -\frac{1}{2} \tan t$.