UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1963

PART II

for Internal and External Students

(8) MATHEMATICS 1

Tuesday 11 June: 10 to 1

Full marks may be obtained for correct answers to about SEVEN questions.

1. If

$$I_n = \int \frac{\sin(2n+1)x}{\sin x}\,dx, \qquad J_n = \int \frac{\sin^2 nx}{\sin^2 x}\,dx,$$

prove that, apart from constants of integration,

$$n(I_n - I_{n-1}) = \sin 2nx,$$

$$J_n - J_{n-1} = I_{n-1}.$$

Evaluate I_2 and J_3 between the limits 0 and $\frac{1}{2}\pi$.

2. (*i*) If $z^2 = xy\,F(x^2 - y^2)$, prove that

$$2xy\left(y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right) = (x^2 + y^2)z.$$

(*ii*) If $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, where α is a constant and if $F(x, y) = \phi(u, v)$, prove that

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}.$$

3. If two skew lines L_1, L_2 have the respective equations

$$\frac{x+3}{4} = \frac{y-3}{-1} = \frac{z-2}{1} \quad \text{and} \quad \frac{x-1}{2} = \frac{y-5}{1} = \frac{z+3}{2},$$

find the direction ratios of a line perpendicular to both. Hence write down the equation of a plane through L_1 parallel to L_2 and show that the shortest distance between the lines is 6.

RP 63/8/10/2/1300

Turn over

4. A triangle ABC in space has vertices whose position vectors are A $(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, B $(\mathbf{i} - 3\mathbf{k})$, C $(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. If O is the origin, calculate

(*i*) the cosine of the angle between $\overrightarrow{OA}$ and $\overrightarrow{BC}$,
(*ii*) the area of the triangle ABC,
(*iii*) the volume of the tetrahedron OABC.

5. A function $f(x)$ is of period 2π and is defined in the interval $0 \leqslant x \leqslant 2\pi$ in the form

$$f(x) = \sin x \qquad 0 \leqslant x \leqslant \pi$$
$$f(x) = 0 \qquad \pi \leqslant x \leqslant 2\pi.$$

Show that

$$f(x) = \frac{1}{\pi} + \tfrac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}.$$

Deduce the sum of the infinite series

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots.$$

6. Solve the following differential equations,
(*i*) $x(x-1)\,dy + (y - x^2 \sin x)\,dx = 0$, given that $y = \tfrac{1}{2}\pi$ when $x = \tfrac{1}{2}\pi$,
(*ii*) $(8y - 4x - 6)\,dy + (2x - 4y + 9)\,dx = 0$, given that $y = 1$ when $x = 0$.

7. The displacement x cm of a point at time t sec satisfies the differential equation

$$\frac{d^2x}{dt^2} + x = 2\cos t.$$

Show that if $x = 0$ and $\frac{dx}{dt} = 1$ when $t = 0$ then $\frac{dx}{dt}$ vanishes when t satisfies the equation

$$t + \tan t = -1.$$

Show graphically that the first positive time at which this occurs is a little greater than $\tfrac{1}{2}\pi$ sec.

8. If x, y, z satisfy the differential equations

$$\frac{dx}{dt} = y + z, \qquad \frac{dy}{dt} = z + x, \qquad \frac{dz}{dt} = x + y,$$

prove that $V = x + y + z$ satisfies $\frac{dV}{dt} = 2V.$

Hence find x, y, z subject to the conditions that $x = 1$, $y = 3$, $z = 2$ when $t = 0$.

9. Assuming that a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \tfrac{1}{4})y = 0$$

can be expressed as a series

$$y = \sum_{r=0}^{\infty} a_r x^{c+r},$$

with $a_0 \neq 0$, show that the possible values of c are $\pm\frac{1}{2}$, and that when $c = \frac{1}{2}$ then

$$a_1 = a_3 = a_5 = \ldots = 0, \quad \text{and} \quad a_{2n} = (-1)^n a_0/(2n+1)!.$$

Show that this series is a multiple of $x^{-\frac{1}{2}} \sin x$.

10. A solid right circular cylinder has mass M, radius r, height h. Prove that its moment of inertia about an axis through its centre perpendicular to its length is

$$M\left(\frac{r^2}{4} + \frac{h^2}{12}\right).$$

A rocket of mass M is in the form of a solid right circular cylinder of radius r, length $8r$, and is travelling in space with velocity u in the direction of its axis. It is struck at the centre of its nose by a meteorite whose mass m is small compared with M, moving with velocity v in the opposite direction at an angle of 30° with the axis of the rocket, and remains embedded in the rocket after impact. Prove that the rocket starts to spin about an axis perpendicular to its length with angular velocity approximately $\dfrac{24mv}{67Mr}$.

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