UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1963

PART II

for Internal and External Students

(8) MATHEMATICS II

Tuesday 11 June: 2.30 to 5.30

Full marks may be obtained for correct answers to about SEVEN questions.

1. (*a*) If $u = e^x \cosh y$ and $v = e^x \sinh y$, verify that

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1,$$

where the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ denotes the determinant $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

(*b*) Find the minimum value of the function

$$\frac{e^{x+2y}}{xy}.$$

2. (*a*) If $y = f(x - ct)$ satisfies the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} + k \frac{\partial y}{\partial t} = 2c^2 \frac{\partial^2 y}{\partial x^2},$$

where c and k are positive constants, show that $f(u)$ satisfies the ordinary differential equation

$$\frac{d^2 f}{du^2} + \frac{k}{c} \frac{df}{du} = 0.$$

Hence find a solution of the partial differential equation which satisfies the conditions
(*i*) $f \to 0$ as $x \to \infty$ for all t,
(*ii*) $f = ae^{kt}$ when $x = 0$.

(*b*) Find the general solution of the ordinary differential equation

$$r^2 \frac{d^2\psi}{dr^2} + 2r \frac{d\psi}{dr} - n(n+1)\psi = 0.$$

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Turn over

3. Show that, for $0 < x < \frac{1}{2}\pi$,

$$\sin x > \frac{2x}{\pi}.$$

Hence show by integration that $1 - \cos x > x^2/\pi$ for $0 < x < \frac{1}{2}\pi$. Show also that

$$e^{-R \sin x} < e^{-2R\,x/\pi} \text{ for } 0 < x < \tfrac{1}{2}\pi,$$

and deduce that.

$$\int_0^{\frac{1}{2}\pi} e^{-R \sin x}\,dx < \frac{\pi}{2R}(1 - e^{-R}), \quad \text{where} \quad R > 0.$$

4. (*i*) Show that the general solution of the equation $\cosh z = -2$ is

$$z = \pm \log(2 + \sqrt{3}) + (2r + 1)j\pi,$$

where $r = 0, \pm 1, \pm 2 \ldots$.

(*ii*) Find all the roots of the equation $z^n = a^n$, where a is real and positive and n is a positive integer, and exhibit these roots on an Argand diagram.

2 **5.** Find the equation of the sphere of radius a, which lies in the positive octant and which touches each of the coordinate planes. Find the coordinates of the centre of the circle in which this sphere is cut by the plane $2x + y + z = 2a$. Find also the equation of the sphere which passes through this circle and the origin.

6. Find the equation of the plane through the lines

$$3x + 2y = 0 = 2y - 3z, \qquad x + y + z = 0 = 4x - y + z.$$ 2.

Find the cosine of the acute angle between this plane and the plane $2x + y + z = 0$ and find also the equations of the line of intersection of these planes in symmetric form.

7. The probability that any one machine will become defective in the small interval between time t and $t + \delta t$ after maintenance is $\alpha e^{-\alpha t}\delta t$, where α is a constant.

A firm possessing 10 similar machines institutes a weekly maintenance of the machines, each of which operates for 44 hours per week, and $\alpha = \frac{1}{400}$ when the unit of time is one hour. Find

(*i*) the probability that all ten machines will continue to function throughout the week,

(*ii*) the probability that more than two machines will become defective before the weekly maintenance is due.

8. Show that the motion of a rigid body about its centre of mass is the same as if the centre of mass were fixed and the same forces act on the body.

A uniform solid rough cylinder of radius a, rotating with angular velocity Ω is placed gently at the foot of a rough plane inclined at an angle β to the horizontal, the direction of rotation being such as to cause it to move up the plane. If the coefficient of friction between the cylinder and the plane is $2 \tan \beta$, find the angular velocity and speed of a point on the axis of the cylinder at time t whilst slipping takes place. Deduce that the cylinder rolls on the plane after a time $a\Omega/(5g \sin \beta)$ and find the angle through which the cylinder has rotated to this instant.

9. Prove, by integration, that the moment of inertia of a uniform solid hemisphere, of mass m and radius a, about a diameter of its base is $2ma^2/5$.
The hemisphere is placed with its curved surface on a rough horizontal plane and slightly disturbed from the equilibrium position. Show that the period of a small oscillation is

$$\frac{2\pi}{5}\sqrt{\left(\frac{26a}{g}\right)}.$$

10. A light spring ABCD, of natural length $3a$ and modulus $4mg$, rests on a smooth horizontal table. The ends A and D are attached to two fixed points on the table distant $3a$ apart. Particles of mass m are attached to the points of trisection B, C of the unstretched string. The system is released from rest with A, B, C, D collinear and with AB $= a$, DC $= 5a/4$. At time t after the start of the motion AB $= a + x$, DC $= a + y$, show that

$$\ddot{x} = -n^2(2x + y), \qquad \ddot{y} = -n^2(x + 2y),$$

where $n^2 = 4g/a$. By subtraction obtain a differential equation for $x - y$. Integrate this equation and hence find x in terms of t.

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