UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1964

PART III

for Internal and External Students

(25) MATHEMATICS I

Wednesday 17 June: 10 to 1

Full marks may be obtained for correct answers to about FIVE of the following fifteen questions.
Candidates may attempt questions from any Section.
More marks will be given for complete answers than for a large number of fragmentary answers.

Section A

Use a separate answer book and write 'SECTION A' on the cover.

1. (*i*) If f is a function of x and y and hence of u and v where

$$u = x + y, \qquad v = \frac{1}{x} + \frac{1}{y},$$

show that

(*a*) $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = u\frac{\partial f}{\partial u} - v\frac{\partial f}{\partial v},$$

(*b*) $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} - \frac{2}{x^2}\frac{\partial^2 f}{\partial u \partial v} + \frac{1}{x^4}\frac{\partial^2 f}{\partial v^2} + \frac{2}{x^3}\frac{\partial f}{\partial v}.$$

(*ii*) Show that the vector $\mathbf{A} = (4xy - z^3)\mathbf{i} + 2x^2\mathbf{j} - 3xz^2\mathbf{k}$ is irrotational and find a function ϕ such that $\mathbf{A} = \text{grad}\ \phi$.

Turn over

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2. If P and Q are functions of the independent variables x and y prove that the necessary and sufficient condition for $P\,dx + Q\,dy$ to be an exact differential is $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Evaluate

$$\int_C (P\,dx + Q\,dy),$$

where $P = 3xy + 4y^2$ and $Q = 2x^2 + 6xy$ and C is the portion of the curve $y = x^2$ between the points O(0, 0) and A(1, 1). Show also that

$$\int_C xy(P\,dx + Q\,dy),$$

where C is a curve through O and A, is independent of the curve and find its value.

3. (*i*) In a tetrahedron OABC the angles BOC, COA, AOB and the sides OA, OB, OC are denoted by α, β, γ and a, b, c respectively. If G is the centroid of the triangle ABC prove that

$$3\vec{OG} = \vec{OA} + \vec{OB} + \vec{OC}$$

and hence that

$$3OG = (a^2 + b^2 + c^2 + 2bc\cos\alpha + 2ca\cos\beta + 2ab\cos\gamma)^{1/2}.$$

(*ii*) (*a*) Prove that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}$.

(*b*) Find the vector $\mathbf{F}$ which satisfies the equations

$$\mathbf{F} \times \mathbf{a} = \mathbf{b}, \qquad \mathbf{F}.\mathbf{c} = p$$

in which p is a given scalar and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are given vectors with $\mathbf{a}.\mathbf{c} \neq 0$.

Section B

Use a separate answer book and write 'SECTION B' on the cover.

4. Assuming a solution of the differential equation

$$x\frac{d^2y}{dx^2} + (4 - x)\frac{dy}{dx} - 2y = 0,$$

in the form of a series $y = x^c(a_0 + a_1x + \ldots + a_rx^r + \ldots)$, show that the possible values of c are 0 and -3, and deduce that the only solution which is such that $y = 1$ when $x = 0$ is

$$y = F(x) = 1 + \frac{x}{2} + \frac{3x^2}{20} + \ldots + \frac{6(r+1)}{(r+3)!}x^r + \ldots$$

By making the substitutions $y = ue^x$, $x = -t$, or otherwise, show that $e^xF(-x)$ is also a solution of the above differential equation and hence that $F(x) = e^xF(-x)$.

5. The Bessel function $J_n(x)$ of integral order n is defined by the generating function

$$\exp\{\tfrac{1}{2}x(t - 1/t)\} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

Prove that $J_{-n}(x) = (-1)^n J_n(x)$.

By using the relationship

$$\exp\{\tfrac{1}{2}x(t - 1/t)\} \exp\{\tfrac{1}{2}y(t - 1/t)\} = \exp\{\tfrac{1}{2}(x + y)(t - 1/t)\}$$

or otherwise, prove that

$$J_n(x + y) = \sum_{r=-\infty}^{\infty} J_r(x)J_{n-r}(y)$$

and deduce that

$$J_0(2x) = \{J_0(x)\}^2 - \{J_1(x)\}^2 + \{J_2(x)\}^2 - \dots$$

6. The coefficients a_r ($r = 0, 1, 2, \dots$) are defined by the relationship

$$(1 - t)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} a_r t^r;$$

prove that

$$a_0 = 1, \qquad a_r = \frac{1.3.5. \dots (2r - 1)}{2.4.6. \dots (2r)}, \qquad r \geqslant 1.$$

Show also that

$$1 - 2t \cos \theta + t^2 = (1 - te^{j\theta})(1 - te^{-j\theta});$$

from this and the relationship

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} t^r P_r(x)$$

deduce that

$$P_{2n+1}(\cos \theta) = 2a_0 a_{2n+1} \cos(2n + 1)\theta + 2a_1 a_{2n} \cos(2n - 1)\theta + \dots + 2a_n a_{n+1} \cos \theta.$$

Hence or otherwise express $\cos 3\theta$ in terms of $P_3(\cos \theta)$ and $P_1(\cos \theta)$.

Section C

Use a separate answer book and write 'SECTION C' on the cover.

7. If $w = f(z)$, where $w = u + jv$ and $z = x + jy$, obtain the Cauchy-Riemann equations connecting the derivatives of u and v with respect to x and y, and deduce that u satisfies Laplace's equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0.$$

In the transformation to parabolic co-ordinates defined by $z = \zeta^2$, where $\zeta = \xi + j\eta$, show that the Laplacian $(\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2)$ becomes

$$\frac{1}{4\rho^2}\left(\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2}\right),$$

where $\rho^2 = \xi^2 + \eta^2$. Sketch the co-ordinate lines ξ = constant, η = constant.

8. State Cauchy's integral formula for an analytic function and use it to integrate the function $(z^2 + 1)/(z^2 - 1)$ along a circle of unit radius with centre at (*a*) $z = 1$, (*b*) $z = -1$.

Use the formula to show that if an analytic function takes known values $u + jv$ on the circle $|z| = R$, its value at an interior point z_0 is

$$u_0 + jv_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{(u + jv)z \, d\theta}{z - z_0}, \quad z = Re^{j\theta}.$$

Prove also that, if z_0 is on the real axis at $z = r$,

$$u_0 = \frac{R}{2\pi}\int_0^{2\pi} \frac{Ru - ru\cos\theta + rv\sin\theta}{R^2 + r^2 - 2Rr\cos\theta} d\theta.$$

9. If the Laplace Transform of $f(t)$ is given by

$$\mathscr{L}f(t) = \int_0^\infty e^{-pt} f(t)\, dt = \bar{f}(p),$$

prove that

$$\mathscr{L}H(t-a)f(t-a) = e^{-ap}\bar{f}(p)$$

where $a \geqslant 0$ and $H(t)$ is the Heaviside unit function defined by

$$H(t) = 0 \text{ for } t < 0$$
$$H(t) = 1 \text{ for } t > 0.$$

Evaluate

(*i*) $f(t) = \mathscr{L}^{-1} \dfrac{1}{p(1 - e^{-p\pi})}$,

(*ii*) $f(t) = \mathscr{L}^{-1} \dfrac{p}{(1 + p^2)(1 - e^{-p\pi})}$.

Sketch a graph of each function.
Solve the differential equation

$$\frac{d^2y}{dt^2} + y = f(t),$$

where $f(t) = n + 1$ for $n\pi < t < (n + 1)\pi$ and given that $y = 0 = \dfrac{dy}{dt}$ when $t = 0$.

Section D

Use a separate answer book and write 'SECTION D' on the cover.

10. Six equal uniform rods, each of length a and of weight W, are freely jointed to form a regular hexagon ABCDEF which rests in equilibrium in a vertical plane with the rod AB in contact with a horizontal plane. Particles each of weight W, are attached to the middle points of CD, DE and EF and the shape of the hexagon is preserved by a light elastic string CF of natural length $3a/2$. Use the Principle of Virtual Work to calculate the modulus of elasticity of the string.

11. A thin chain of weight W and of length $2l$, has one end fixed at A, and the other end is attached to a small ring, of weight nW, which can slide on a fixed rough horizontal rod passing through A. Show that the greatest distance of the ring from A is

$$\frac{2l}{\rho}\log[\rho + \sqrt{(1 + \rho^2)}],$$

where ρ is given in terms of μ the coefficient of friction between the ring and the rod by the equation $(2n + 1)\rho\mu = 1$.

12. A uniform disc of mass $2m$ and radius a is free to rotate about a horizontal axis through the centre of the disc and perpendicular to its plane. One end of a light inextensible string of length $2a$ is attached to the disc at a point on its circumference and a particle of mass m is attached to the other end of the string. The system oscillates about the position of stable equilibrium. Use Lagrange's Equations to find the periods of the normal modes.

Section E

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13. Show that the latent vectors of a real symmetric matrix, corresponding to distinct latent roots, are orthogonal.

The 3×3 symmetric matrix A has latent roots 3, 6, −9. The vector $\{-2, 2, 1\}$ corresponds to the root 3 and the vector $\{1, 2, -2\}$ corresponds to the root −9. Find a latent vector corresponding to the root 6 and calculate A from the relation $A = M\Lambda M^{-1}$ where M is a matrix having the latent vectors as its columns and Λ is the diagonal matrix of latent roots.

14. Define the operators E, D, Δ and show that $E = e^{hD} = 1 + \Delta$, where h is the interval of tabulation.

Hence obtain the formulae

$$hf_0' = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \dots)f_0$$
$$= \tfrac{1}{2}(f_1 - f_{-1}) - \tfrac{1}{6}\Delta^3 f_{-1} + \dots$$

Derive a formula for f_0'' and hence calculate $f'(2)$ and $f''(2)$ for the third degree polynomial given by:

x	0	1	2	3	4
f	2	2	8	26	62

dy/dx

Δ

x y_{n-1} y_n y_{n+1}

Turn over

15. A function $f(x)$ is given at points $x_n = x_0 + nh$. If $x - x_0 = ph$ show that the parabola $y = \frac{1}{2}(p+1)(p+2)f_0 - p(p+2)f_{-1} + \frac{1}{2}p(p+1)f_{-2}$ passes through three successive table points.

Hence derive the approximate integration formulae

$$\int_{x_0}^{x_1} f(x)\,dx = \tfrac{1}{12}h(23f_0 - 16f_{-1} + 5f_{-2}),$$

$$\int_{x_{-1}}^{x_0} f(x)\,dx = \tfrac{1}{12}h(5f_0 + 8f_{-1} - f_{-2}).$$

Use these formulae to carry the solution of the differential equation

$$\frac{dy}{dx} = \frac{2x-1}{x^2}y + 1$$

one step forward if the values given below have been calculated already.

x	1	1·1	1·2
y	2	2·3148	2·6589
$\frac{dy}{dx}$	3	3·2957	3·5851

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