UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1964

PART II

for Internal and External Students

(8) MATHEMATICS I

Monday 8 June: 10 to 1

Full marks may be obtained for correct answers to about SIX questions.

1. Show that

(*a*)
$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

(*b*)
$$\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx$$
$$= \frac{1}{2}\int_0^{\pi/2} \log\sin 2x\,dx - \frac{\pi}{4}\log 2.$$

Deduce that

$$\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2.$$

2. The position vectors of the vertices of a triangle ABC are respectively

$$\mathrm{A}\ (\mathbf{i}-\mathbf{k}),\qquad \mathrm{B}\ (2\mathbf{i}+\mathbf{j}+3\mathbf{k}),\qquad \mathrm{C}\ (3\mathbf{i}+2\mathbf{j}+\mathbf{k}),$$

where **i**, **j**, **k**, are mutually perpendicular unit vectors. Show that the angle ACB is a right angle and find cos BAC.

Calculate the area of the triangle ABC.

Turn over

RP 64/193/9/2/1300

3. (*i*) Solve the equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0,$$

given that $y = 2$ and $\frac{dy}{dx} = 1$ when $x = 0$.

(*ii*) Solve the equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 36x,$$

given that $y = 0$ and $\frac{dy}{dx} = 4$ when $x = 0$.

Find, for this solution, the value of x for which $\frac{d^2y}{dx^2} = 0$.

4. Find the components of a unit vector normal to the surface

$$3x^2 + y^2 + 2z^2 - 4yz + zx + 5xy = 1$$

at either of the points P, Q where it is intersected by a straight line through the origin with direction cosines $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$.

Also show that the equations of the tangent planes at P, Q are

$$2x + 7y - 17z \pm 2\sqrt{23} = 0.$$

5. Obtain an expansion, valid for $0 < x < \pi$, for the function $y = x^2$ of the form

$$\sum_{n=0}^{\infty} a_n \cos nx.$$

Find the sum of this series (*a*) when $x = \frac{\pi}{4}$ and (*b*) when $x = \frac{5\pi}{4}$.

Show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}.$$

6. Sketch graphs of $e^x - 1$ and $\log(x + 2)$, and show that the equation

$$e^x - 1 = \log(x + 2)$$

has a root between 0·5 and 1. By Newton's method, or otherwise, calculate this root correct to two places of decimals.

7. Allowing for errors in observation the following pairs of numbers are connected by a relation of the form $y = ka^x$ where k and a are constants:

x	1·0	1·2	1·4	1·6
y	11·8	14·6	17·9	22·1

Use the method of least squares to find the best values of k and a.

8. Assuming a series solution of the differential equation

$$4x\frac{d^2y}{dx^2} + (4x+3)\frac{dy}{dx} + y = 0$$

of the form $a_0x^c + a_1x^{c+1} + \ldots + a_nx^{c+n} + \ldots$, show that the two possible values of c are 0 and $\frac{1}{4}$.

When $c = 0$ show that

$$(4n+1)a_n + (n+1)(4n+3)a_{n+1} = 0$$

and write down the first four terms of the series for this value of c.

9. A particle of mass m rests on a horizontal table attached to one end of a horizontal spring of negligible weight, the other end of which is fixed. When the spring is extended or compressed the force is $m\omega^2$ times the change in its length, ω^2 being constant. The coefficient of friction between the particle and the table is μ.

The particle is struck a blow of impulse mV along the line of the spring so as to extend the spring. Show that the distance the particle has travelled when it first comes to rest is

$$\frac{1}{\omega^2}\left\{(\mu^2g^2 + V^2\omega^2)^{1/2} - \mu g\right\}.$$

10. An inextensible light string ABCD has its end A fixed. It passes underneath a pulley which rests on the string with the portion BC of the string in contact with the pulley. AB hangs vertically and CD is pulled vertically upwards by a force P. The string does not slip on the pulley but the pulley turns freely on its axis. The pulley is of mass M and radius a, and its radius of gyration k satisfies $k^2 = \frac{3}{4}a^2$. A load of mass $2M$ is suspended from the pulley. If $P = 2Mg$ show, by using the energy equation or otherwise, that when the pulley has risen a distance x from rest its velocity v is given by $v^2 = \frac{8}{15}gx$.

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