UNIVERSITY OF LONDON

B.Sc. (ENGINEERING) EXAMINATION 1964

PART II

for Internal and External Students

(8) MATHEMATICS II

Monday 8 June: 2.30 to 5.30

Full marks may be obtained for correct answers to about SIX questions.

1. Given that U and V are functions of the independent variables r and θ such that

$$\frac{\partial V}{\partial r} = -\sin\theta\,\frac{\partial U}{\partial \theta} \quad \text{and} \quad \frac{\partial V}{\partial \theta} = r^2 \sin\theta\,\frac{\partial U}{\partial r},$$

prove that

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial U}{\partial \theta}\right) = 0$$

and

$$r^2 \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2} - \cot\theta\,\frac{\partial V}{\partial \theta} = 0.$$

If the function V is independent of r, find the general value of V in terms of θ.

2. Show that the equation

$$\frac{\partial z}{\partial t} = a^2 \frac{\partial^2 z}{\partial x^2}$$

has a solution of the form $z = T(t)\sin(px + \alpha)$, where $T(t)$ is a function of t only.

Obtain a solution of the equation subject to the following conditions:

(*i*) $z = 0$ when $x = 0$ and when $x = a$, for all values of t,

(*ii*) $z = 3\sin\dfrac{2\pi x}{a}$ when $t = 0$, for all values of x.

7

Turn over

RP 64/131/8/2/1300

$(x^2+y^2+z^2) - \frac{12}{9}(x+y+z) +$

$\frac{12}{9} = 2a$

$a = \frac{12}{18} = \frac{4}{6}\ \frac{2}{3}$

2. $\frac{16}{81}$ $\frac{32}{81}$.

3. (*i*) By differentiating

$$\frac{Ax + B}{x^2 + 2x + 5} + C \tan^{-1} \frac{x + 1}{2},$$

or otherwise, evaluate

$$\int_{-1}^{1} \frac{x \, dx}{(x^2 + 2x + 5)^2}.$$

(*ii*) Find the maximum value of $V = xyz$ subject to the condition that $x + y + 2z = 3a$, where a is a constant.

$5a = 15 + 1$

$a = \frac{3}{4}$ 4.

4. Prove that

$$|\sinh(x + jy)| = \sqrt{\{\sinh^2 x + \sin^2 y\}}.$$

If two complex variables $z = x + jy$ and $w = u + jv$ are connected by the equation

$$w = \log \operatorname{cosech} \frac{\pi z}{a},$$

where a is a real positive constant, show that

(*a*) $\sinh^2 \dfrac{\pi x}{a} + \sin^2 \dfrac{\pi y}{a} = e^{-2u},$

(*b*) $\coth \dfrac{\pi x}{a} \tan \dfrac{\pi y}{a} = -\tan v.$

Deduce that if u is large and positive, then $x^2 + y^2$ is approximately equal to $a^2 \pi^{-2} e^{-2u}$.

5. A sphere has its centre within the tetrahedron determined by the planes $x = 0$, $y = 0$, $z = 0$ and the plane P whose equation is $2x + 3y + 6z = 12$. The sphere touches all four of these planes; show that its equation is

$$9(x^2 + y^2 + z^2) - 12(x + y + z) + 8 = 0,$$

and find the coordinates of the point of contact of the sphere with the plane P.

Find also the equation of the other tangent plane to the sphere which is parallel to P.

6. The straight lines L_1, L_2 both pass through the origin in three-dimensional space and have direction ratios $2 : 3 : 6$ and $3 : 6 : 2$ respectively. Show that the equations of the two lines in the plane $x = 0$ which are equally inclined to L_1 and L_2 are

$$x = 0,\ 3y - 4z = 0 \text{ and } x = 0,\ 9y + 8z = 0.$$

Find also the equations of the two lines in the plane $z = 0$ which are equally inclined to L_1 and L_2, and hence find the equation of the plane which bisects the acute angle between L_1 and L_2.

7. A machine is powered by three similar storage batteries; it will function satisfactorily only if at least two of these batteries are serviceable. The probability of any one battery becoming unserviceable in less than 50 hours is 0·2, and of becoming unserviceable in less than 100 hours is 0·6.

Find the probability that the machine will function satisfactorily for (*a*) at least 50 hours, (*b*) between 50 and 100 hours.

8. A wheel and axle of total mass m is placed with the axle on and perpendicular to a pair of rough parallel rails in a horizontal plane so that the wheel is between the rails. The radius of the axle is b and of the wheel $2b$, whilst the moment of inertia of the combined wheel and axle about its axis is $2mb^2$. The wheel is rotating with angular velocity ω when it is placed on the rails and is immediately acted upon by a constant couple $\frac{1}{2}mbg$ in a plane perpendicular to the axle and in the opposite sense to ω. The couple continues to act throughout the ensuing motion. If the coefficient of friction between the rails and the axle is $\frac{1}{2}$ find the time which elapses until the system begins to roll without slipping on the rails. Find also the linear velocity of the wheel at that time.

Show that, in the rolling motion which follows, the wheel and axle comes instantaneously to rest after a *total* time of $4b\omega/g$.

9. A light rigid thin circular wire of radius a and centre O has three particles each of mass m attached to it at points A, B and C such that the arcs AB and BC subtend right angles at the centre, A and C being at opposite ends of a diameter. The wire is placed in a vertical plane with the point C in contact with a rough horizontal plane and released from rest. Show that if P is the point of contact at a subsequent time t and the angle POB $= \theta$, then

$$a(3 - \cos\theta)\dot{\theta}^2 = g\cos\theta.$$

$a -$

Show also that the vertical reaction at the point of contact when the system is in the symmetrical position is $7mg/2$.

$(a - a\cos\theta.)$

10. AB and BC are two light elastic strings both of modulus $3mg$ and of natural lengths a and $2a$ respectively. A particle of mass m is attached to B whilst the ends A and C are attached to fixed points on a smooth horizontal table so that ABC is a straight line with AB $= 3a/2$ and BC $= 3a$. The mass m is displaced along the line ABC through a distance $\frac{1}{2}a$ towards C and released from rest. Find the velocity v_1 of m as it passes through the original position of B.

$a - (a - a\cos$

$= \cos\theta.$

If the strings are now placed in a vertical position with A uppermost and AC $= 9a/2$, find the position of equilibrium of the particle. The particle is moved to the same displaced position as before (distant $2a$ from A) and again released from rest. Show that the velocity v_2 of the mass as it passes through the *original* position of B, where AB $= \dfrac{3a}{2}$, is given by

$$v_2{}^2 = v_1{}^2 - ga.$$

F. M. ARSCOTT
B. A. EDWARDS
D. E. R. GODFREY

$V =$

$v = r\omega$

$v = a\dot{\theta}$